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Convergence analysis of the rectangular Morley element scheme for second order problem in arbitrary dimensions

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Abstract In this paper, we present the convergence analysis of the rectangular Morley element scheme utilised on the second order problem in arbitrary dimensions. Specifically, we prove that the convergence of the scheme is of $\mathcal{O}(h)$ order in energy norm and of $\mathcal{O}(h^2)$ order in L^2 norm on general d-rectangular triangulations. Moreover, when the triangulation is uniform, the convergence rate can be of $\mathcal{O}(h^2)$ order in energy norm, and the convergence rate in L^2 norm is still of $\mathcal{O}(h^2)$ order, which can not be improved. Numerical examples are presented to demonstrate our theoretical results.

Keywords d-rectangular Morley element second order elliptic equation convergence analysis super convergence lower bound estimate

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1 Introduction

In applied sciences, many model problems take the formulation of fourth order elliptic perturbation problems, such as, e.g., the linearized Cahn-Hilliard equation [9, 31, 39, 43, 51–53], and the strain gradient problem [1, 10, 11, 28, 33, 49]. In order for the robust discretisation of such problems, numerical schemes that work for both fourth and second order problems are needed. The rectangular Morley element (RM element for short in the sequel) scheme is one that falls into this category. The RM element is introduced by Wang, Shi and Xu ([40], 2007) originally for fourth order problem in arbitrary dimension; its *a priori* ([40]), *a posteriori* ([4]) and superconvergence ([18]) analysis have been established already. Utilised for second order problems, the RM element scheme is of nonconforming type, and its validity for second order problems has been pointed out in Shi-Wang [37], however, without a formal statement and technical proof. In this paper, we will present a technical and complete analysis of the RM element scheme utilised on the second order problem in arbitrary dimension. Specifically, for the energy norm of the error, beside the standard analysis which lead to a convergence of the scheme which is of $\mathcal{O}(h)$ order on general shape regular triangulations, a more careful analysis is given; namely, when the triangulation is divisionally uniform (see its precise description in Section 3), the convergence rate can be of $\mathcal{O}(h^{1.5})$ order, and when the triangulation is uniform, the convergence rate is $\mathcal{O}(h^2)$. For the L^2 norm, an $\mathcal{O}(h^2)$ convergence rate

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can be obtained for convex domain; and in general, this estimate can not be improved. The discussion on these uniform triangulations illustrates the convergence analysis being a sharp one.

As the RM element is of nonconforming type, revealed by the Strang lemma, work has to be spent on the analysis of the consistency error. It is well known that once the zero-th order or first order moment of the finite element function is continuous across the internal interfaces of the grids, the consistency error of first order (see, e.g., [15, 19, 20, 24, 29, 32]) or of second order (see, e.g., [22, 44]) can be proved. Besides, based on the symmetry of the rectangular cells, another standard way can be to constructing an internal orthogonal space on every cell, which can lead to a consistency error of first order (see, e.g., the Wilson element) or second order (see, e.g., [21, 27]) by the internal-eliminating technique. However, neither of the techniques above works for the RM element directly. Actually, the average of the RM element function is not continuous across the internal interfaces, and a direct utilization of the internal-eliminating technique for the RM element can lead to a consistency error of first order only, which can not explain the high accuracy of the scheme on uniform grids. By the aid of stable decomposition, we can indeed divide the consistency analysis on the whole finite element space to an equivalent system of subproblems on a big subspace associated with vertex degree of freedoms (DOFs) and a series of smaller subspaces each associated with a face DOF. We can thus implement the internal-eliminating technique with respect to cells for the big and small subspaces, and generalize the internal-eliminating technique onto patches for every small subspace, and arrive at the convergence rate in energy norm with respect to various triangulations, namely $\mathcal{O}(h)$ on general grids and $\mathcal{O}(h^2)$ for uniform grids. The application of duality argument seems standard, and an error estimate of $\mathcal{O}(h^2)$ is achieved in L^2 norm. However, we note that people can not find a nontrivial conforming subspace in the RM element space; the bilinear element space, e.g., is not contained in. This implies that we can not expect the L^2 norm of the error being one order higher than the energy norm of the error. Indeed, we prove rigorously for uniform triangulations that the L^2 norm of error can not be nontrivially higher than $\mathcal{O}(h^2)$. All the analysis above is carried out in a unified form for arbitrary dimensions, and numerical experiments confirm the theoretical results.

We remark that, a key fact of our analysis is higher consistency accuracy can be expected on uniform triangulations. This fact is studied in the context of superconvergence. Also, this higher accuracy analysis has been a basis of further study of the patch recovery technique and *a posteriori* error analysis. We refer to [7, 8, 12, 14, 23, 25, 36, 55–59] for related discussions. By the general theory of fast auxiliary space preconditioning (FASP) (c.f. [13, 46, 47, 54]), the stable decomposition will also play a fundamental role in designing optimal preconditions in future.

In the sequel, we will use the following standard notation. We use Ω for a general bounded polyhedral domain in $\mathbb{R}^d(d \geqslant 2)$, $\partial\Omega$ the boundary of Ω, and $\mathbf{n} = \Big(n_1, n_2, \cdots, n_d\Big)^{\mathrm{T}}$ the unit outer normal to $\partial\Omega$. For a nonnegative integer s, we shall use the usual Sobolev spaces such as $H^s(K)$ with the corresponding seminorm and norm denoted by $|\cdot|_{s,K}$ and $||\cdot||_{s,K}$, respectively. $(\cdot,\cdot)_K$ denotes the inner product of $L^2(K)$. When $K = \Omega$, we just write $|\cdot|_s$, $||\cdot||_s$ and $(\cdot,\cdot)$. Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$, set $|\alpha| = \sum_{i=1}^d \alpha_i$ and $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \forall x \in \mathbb{R}^d$. For a subset $B \subset \mathbb{R}^d$ and a nonnegative integer r, Let $P_r(B)$ and $Q_r(B)$ be the spaces of polynomials on B defined by$P_r(B) = \mathrm{span}\{x^\alpha||\alpha| \leqslant r\}, \quad Q_r(B) = \mathrm{span}\{x^\alpha|\alpha_i \leqslant r, 1 \leqslant i \leqslant d\}$. In this paper, we use C to denote a generic positive constant which may be different at different places. Also, following [45], $\lesssim$, $\gtrsim$ and $\eqsim$ denote $\leqslant$, $\geqslant$ and $=$ up to a constant, respectively. The hidden constants depend on the domain, and, they also depend on the shape-regularity of the triangulation when it is involved, but they do not depend on h or any other mesh parameter.

The remaining of the paper is organized as follows. In Section 2, we present some preliminaries of the RM element in any dimension. In Section 3, we study the discretisation scheme of the model problem, and we construct the error estimates in energy norm and L^2 norm on general d-rectangular triangulations. In Section 4, some numerical examples are presented to demonstrate our theoretical results. Finally, in Section 5, some conclusions are given.

2 Preliminaries

2.1 The d-rectangular Morley element

Let $K \subset \mathbb{R}^d$ be a d-rectangle, $x_c = (x_{1,c}, \cdots, x_{i,c}, \cdots, x_{d,c})^{\mathrm{T}} \in \mathbb{R}^d$ be the barycenter of K, and h_i the half length of K in x_i direction, $i = 1, 2, \ldots, d$. Then the d-rectangle can be denoted by

$$K = \{x = (x_1, \cdots, x_i, \cdots, x_d)^{\mathrm{T}} \mid x_i = x_{i,c} + \xi_i h_i,\ -1 \leqslant \xi_i \leqslant 1,\ 1 \leqslant i \leqslant d\}, \tag{2.1}$$

and the vertices a_i, $1 \leqslant i \leqslant 2^d$, of K are denoted by

$$a_i = (x_{1,c} + \xi_{i1} h_1, \cdots, x_{j,c} + \xi_{jd} h_j, \cdots, x_{d,c} + \xi_{id} h_d)^{\mathrm{T}},\ |\xi_{ij}| = 1,\ 1 \leqslant j \leqslant d,\ 1 \leqslant i \leqslant 2^d.$$

Moreover, let F_{2j-1} and F_{2j} $(1 \leqslant j \leqslant d)$ denote the two $(d-1)$-faces of K perpendicular to x_j axe as

$$F_{2j-1} = \{x = (x_1, \cdots, x_i, \cdots, x_d)^{\mathrm{T}} \mid x_i = x_{i,c} + \xi_i h_i,\ -1 \leqslant \xi_i \leqslant 1,\ 1 \leqslant i \leqslant d,\ i \neq j,\ \xi_j = 1\},$$

$$\text{and } F_{2j} = \{x = (x_1, \cdots, x_i, \cdots, x_d)^{\mathrm{T}} \mid x_i = x_{i,c} + \xi_i h_i,\ -1 \leqslant \xi_i \leqslant 1,\ 1 \leqslant i \leqslant d,\ i \neq j,\ \xi_j = -1\}.$$

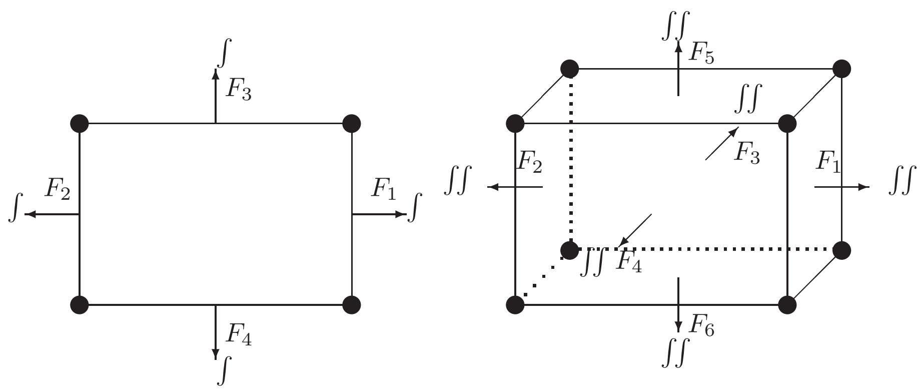

Figure 1 Degrees of freedom of the rectangular Morley element in two and three dimensions.

The d-rectangular Morley element ([37,40]) is defined by the triple $(K, P_M(K), D)$, where

- the geometric shape K is a d-rectangle;
- the shape function space is

$$P_M(K) := Q_1(K) + \mathrm{span}\{x_i^2, x_i^3 \mid 1 \leqslant i \leqslant d\}; \tag{2.2}$$

- the vector $D(v)$ of degrees of freedom is, for any $v \in C^1(K)$(see Figure 1),

$$D(v) := \left(v(a_i),\ \frac{1}{|F_j|}\int_{F_j} \frac{\partial v}{\partial \mathbf{n}_{F_j}} ds\right),\ \ i = 1, \ldots, 2^d,\ j = 1, \ldots, 2d, \tag{2.3}$$

where $\mathbf{n}_{F_j}$ is the unit normal vector of $(d-1)$-face F_j, and $|F_j|$ denotes measure of $(d-1)$-face F_j.

The triple is P_K-unisolvent. Indeed, define

$$\begin{cases} p_i = \dfrac{1}{2^{d+1}}(2\displaystyle\prod_{j=1}^{d}(1 + \xi_{ij}\frac{x_j - x_{j,c}}{h_j}) - \sum_{j=1}^{d}\xi_{ij}\frac{x_j - x_{j,c}}{h_j}((\frac{x_j - x_{j,c}}{h_j})^2 - 1)), & 1 \leqslant i \leqslant 2^d, \\ q_{2k-1} = \dfrac{h_k}{4}(\dfrac{x_k - x_{k,c}}{h_k} + 1)^2(\dfrac{x_k - x_{k,c}}{h_k} - 1), & 1 \leqslant k \leqslant d, \\ q_{2k} = -\dfrac{h_k}{4}(\dfrac{x_k - x_{k,c}}{h_k} + 1)(\dfrac{x_k - x_{k,c}}{h_k} - 1)^2, & 1 \leqslant k \leqslant d. \end{cases} \tag{2.4}$$

Then it can be verified that, with δ_{ij} the Kronecker symbol,

$$\begin{cases} p_i(a_j)=\delta_{ij}, & 1\leqslant i,j\leqslant 2^d,\\ \dfrac{1}{|F_j|}\displaystyle\int_{F_j}\frac{\partial p_i}{\partial \mathbf{n}_{F_j}}ds=0, & 1\leqslant i\leqslant 2^d, 1\leqslant j\leqslant 2d;\\ q_i(a_j)=0, & 1\leqslant i\leqslant 2d, 1\leqslant j\leqslant 2^d,\\ \dfrac{1}{|F_j|}\displaystyle\int_{F_j}\frac{\partial q_i}{\partial \mathbf{n}_{F_j}}ds=\delta_{ij}, & 1\leqslant i,j\leqslant 2d. \end{cases} \tag{2.5}$$

The corresponding interpolation operator Π_K is then given by

$$\Pi_K v=\sum_{i=1}^{2^d}p_i v(a_i)+\sum_{j=1}^{2d}q_j\frac{1}{|F_j|}\int_{F_j}\frac{\partial v}{\partial \mathbf{n}_{F_j}}ds, \forall v\in C^1(K).$$

Denote $P_M^{\mathcal{X}}(K)=\operatorname{span}\{p_i, 1\leqslant i\leqslant 2^d\}$, $P_M^{\mathcal{F}}(K)=\operatorname{span}\{q_i, 1\leqslant i\leqslant 2d\}$, and $Q_M^k(K)=\operatorname{span}\{q_{2k-1},q_{2k}\}, 1\leqslant k\leqslant d$. Besides, for f a $(d-1)$-face of K, denote

$$P_M^f(K):=\{v\in P_M(K): v \text{ vanishes on vertices of } K,\\ \int_{f'}\frac{\partial v}{\partial \mathbf{n}_{f'}}ds \text{ vanishes on any face } f'\subset\partial K \text{ other than } f\}.$$

Further, given f a $(d-1)$-face, we can construct two d-rectangles K_L and K_R that share f as a common face. Denote $\omega_f:=K_L\cup K_R$ as the patch associated with f. $|K_L|$ and $|K_R|$ denote the measure of elements K_L and K_R, respectively. If $|K_L|=|K_R|$, then the patch is called uniform. Define

$$P_M^f(\omega_f):=\{v_h\in L^2(\Omega): v_h|_K\in P_M^f(K),\ K\in\{K_L,K_R\},\ \int_f\frac{\partial v_h}{\partial \mathbf{n}_f}ds \text{ is continuous on } f\}.$$

Evidently, $\dim(P_M^f(K))=1$.

2.2 Structural properties of the shape function space

2.2.1 Local orthogonality of the finite element function

As the foundation of the theoretical analysis, some facts on local orthogonality or near orthogonality have to be revealed. By direct calculation, we obtain Lemmas 2.1 and 2.2 below. On every element K, let Π_K^1 be the nodal interpolation operator associated with $Q_1(K)$ element.

Lemma 2.1. Let K be a d-rectangle, $\mathbf{n}^K$ be the outside normal direction of ∂K and n_i^K be the i-th component of $\mathbf{n}^K, i=1,\ldots,d$. Then

1. it holds for $\phi\in P_M^{\mathcal{X}}(K)$ and $p_1^K\in P_1(K)$ that

$$\int_{\partial K}p_1^K(\phi-\Pi_K^1\phi)n_i^K ds=0,\quad 1\leqslant i\leqslant d; \tag{2.6}$$

2. it holds for $\psi\in P_M^{\mathcal{F}}(K)$ that

$$\int_{\partial K}\psi n_i^K ds=0,\quad 1\leqslant i\leqslant d. \tag{2.7}$$

Proof. Let K be denoted as (2.1). In order to simplify the presentation, denote $\xi_i=\frac{x_i-x_{i,c}}{h_i}, 1\leqslant i\leqslant d$.

1. For any $\phi\in P_M^{\mathcal{X}}(K)$, we have $\phi-\Pi_K^1\phi\in\operatorname{span}\{\xi_j(\xi_j^2-1), 1\leqslant j\leqslant d\}$. It can be verified that

$$\int_{F_{2i-1}}\xi_k\xi_j(\xi_j^2-1)ds=\int_{F_{2i}}\xi_k\xi_j(\xi_j^2-1)ds=0,\quad 1\leqslant i\leqslant d,\quad 0\leqslant k\neq j\leqslant d,$$

where $\xi_0 := 1$. Now given $p_1^K = c_0 + c_1\xi_1 + \cdots + c_d\xi_d$, thus $p_1^K|_{F_{2i-1}} = c_i - c_i\xi_i + \sum_{k=0}^d c_k\xi_k$ and $p_1^K|_{F_{2i}} = -c_i - c_i\xi_i + \sum_{k=0}^d c_k\xi_k$, we have, for $i = 1, \ldots, d$,

$$\begin{aligned}\int_{\partial K} p_1^K \xi_j(\xi_j^2-1)n_i^K ds &= \int_{F_{2i-1}} p_1^K \xi_j(\xi_j^2-1)ds - \int_{F_{2i}} p_1^K \xi_j(\xi_j^2-1)ds \\ &= \int_{F_{2i-1}} (c_j\xi_j - c_i\xi_i)\xi_j(\xi_j^2-1)ds - \int_{F_{2i}} (c_j\xi_j - c_i\xi_i)\xi_j(\xi_j^2-1)ds \\ &= \begin{cases} \displaystyle\int_{F_{2i-1}} (c_j\xi_j - c_j\xi_j)\xi_j(\xi_j^2-1)ds - \int_{F_{2i}} (c_j\xi_j - c_j\xi_j)\xi_j(\xi_j^2-1)ds = 0, & \text{if } i = j, \\ \displaystyle\int_{F_{2i-1}} c_j\xi_j^2(\xi_j^2-1)ds - \int_{F_{2i}} c_j\xi_j^2(\xi_j^2-1)ds = 0, & \text{if } i \neq j. \end{cases}\end{aligned} \tag{2.8}$$

2. Let $q \in P_M^{\mathcal{F}}(K)$, then $q|_{F_{2i-1}} = q|_{F_{2i}}$ as two functions of $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$, $i = 1, \ldots, d$. Therefore, for $i = 1, \ldots, d$,

$$\int_{\partial K} q n_i^K ds = \int_{F_{2i-1}} q ds - \int_{F_{2i}} q ds = 0. \tag{2.9}$$

The proof is finished. □

Lemma 2.2. Let f be a $(d-1)$-face, ω_f be the patch associated with f, $\mathbf{n}^{\omega_f}$ be the outside normal direction of $\partial\omega_f$, and $n_i^{\omega_f}$ be the i-th component of $\mathbf{n}^{\omega_f}$. If ω_f is uniform, then it holds for $\psi \in P_M^f(\omega_f)$, $p_1^{\omega_f} \in P_1(\omega_f)$ that

$$\int_{\partial\omega_f} p_1^{\omega_f}\psi n_i^{\omega_f} ds = 0, \quad i = 1, ..., d. \tag{2.10}$$

Proof. Let the barycenter of $(d-1)$-face f be $(f_{1,c}, \cdots, f_{k,c}, \cdots, f_{d,c})$ and the half length of K_L and K_R in x_i direction be h_i , $i = 1, 2, \ldots, d$. Then, with x_j being orthogonal to f, we can denote

$$f = \{x = (x_1, \cdots, x_k, \cdots, x_d)^{\mathrm{T}} \,|\, x_k = f_{k,c} + \xi_k h_k,\ -1 \leqslant \xi_k \leqslant 1,\ 1 \leqslant k \leqslant d,\ k \neq j,\ \xi_j = 0\},$$

$$K_L = \{x = (x_1, \cdots, x_k, \cdots, x_d)^{\mathrm{T}} \,|\, x_k = f_{k,c} + \xi_k h_k,\ -1 \leqslant \xi_k \leqslant 1,\ 1 \leqslant k \leqslant d,\ k \neq j,\ -2 \leqslant \xi_j \leqslant 0\},$$

and

$$K_R = \{x = (x_1, \cdots, x_k, \cdots, x_d)^{\mathrm{T}} \,|\, x_k = f_{k,c} + \xi_k h_k,\ -1 \leqslant \xi_k \leqslant 1,\ 1 \leqslant k \leqslant d,\ k \neq j,\ 0 \leqslant \xi_j \leqslant 2\}.$$

Now, without loss of generality, let $p_1^{\omega_f} = c_0 + c_1x_1 + \cdots + c_dx_d$ on ω_f and $\psi \in P_M^f(\omega_f)$, namely,

$$\psi = \begin{cases} \alpha\dfrac{h_j}{4}(\dfrac{x_j - (f_{j,c} - h_j)}{h_j} + 1)^2(\dfrac{x_j - (f_{j,c} - h_j)}{h_j} - 1) & \text{on } K_L, \\ -\alpha\dfrac{h_j}{4}(\dfrac{x_j - (f_{j,c} + h_j)}{h_j} + 1)(\dfrac{x_j - (f_{j,c} + h_j)}{h_j} - 1)^2 & \text{on } K_R, \end{cases}$$

with some $\alpha \in \mathbb{R}$. Elementary calculus leads to that

$$\int_{\partial K_L} p_1^{\omega_f}\psi n_i^{\omega_f} ds = \begin{cases} 0, & \text{if } j = i, \\ -\frac{1}{6}c_ih_j\alpha|K_L|, & \text{otherwise;} \end{cases} \quad \text{and,} \int_{\partial K_R} p_1^{\omega_f}\psi n_i^{\omega_f} ds = \begin{cases} 0, & \text{if } j = i, \\ \frac{1}{6}c_ih_j\alpha|K_R|, & \text{otherwise.} \end{cases}$$

Since $p_1^{\omega_f}$ and ψ are both continuous across f, ω_f is uniform and thus $|K_L| = |K_R|$, we have

$$\int_{\partial\omega_f} p_1^{\omega_f}\psi n_i^{\omega_f} ds = \int_{\partial K_L} p_1^{\omega_f}\psi n_i^{\omega_f} ds + \int_{\partial K_R} p_1^{\omega_f}\psi n_i^{\omega_f} ds = 0, \quad i = 1, \ldots, d.$$

This finishes the proof. □

Lemma 2.3. There exists a constant $\theta \in (0, \frac{1}{2})$ depending on d only, such that, for any K a d-rectangle,

1. it holds for nodal basis function $q_i, q_j (1 \leqslant i \neq j \leqslant 2d)$, that
$$|(\nabla q_i, \nabla q_j)_K| \leqslant \theta(||\nabla q_i||_{0,K}^2 + ||\nabla q_j||_{0,K}^2); \tag{2.11}$$
2. it holds for any $\phi \in P_M^{\mathcal{X}}(K)$ and $\psi \in P_M^{\mathcal{F}}(K)$ that
$$|(\nabla \phi, \nabla \psi)_K| \leqslant \theta(||\nabla \phi||_{0,K}^2 + ||\nabla \psi||_{0,K}^2). \tag{2.12}$$

Proof. Firstly, direct calculation leads to that, with $i \neq j$,
$$\frac{|(\nabla q_i, \nabla q_j)_K|}{\|\nabla q_i\|_{0,K}^2 + \|\nabla q_j\|_{0,K}^2} = \begin{cases} \frac{1}{8}, & \text{if } \{i,j\} = \{2k-1, 2k\} \text{ for some } k,\ 1 \leqslant k \leqslant d; \\ 0, & \text{otherwise.} \end{cases}$$
Thus (2.11) is proved.

Secondly, according to the definition, $P_M^{\mathcal{F}}(K)$ can be decomposed as
$$P_M^{\mathcal{F}}(K) = Q_M^1(K) \oplus \cdots \oplus Q_M^d(K). \tag{2.13}$$
Moreover, the decomposition is orthogonal with respect to the inner product $(\nabla \cdot, \nabla \cdot)_K$. Actually, $\partial_i q = 0$ for $q \in Q_M^j(K)$ with $i \neq j$. Further, for every k, $1 \leqslant k \leqslant d$, $Q_M^k(K)$ can be decomposed as
$$Q_M^k(K) = \mathrm{span}\{q_{2k-1}, q_{2k}\} = \mathrm{span}\{q_{2k-1} + q_{2k}\} \oplus \mathrm{span}\{q_{2k-1} - q_{2k}\}. \tag{2.14}$$
Where $(q_{2k-1} + q_{2k})$ is perpendicular to $P_M^{\mathcal{X}}(K)$, and this decomposition (2.14) is orthogonal with respect to both the inner products $(\partial_k \cdot, \partial_k \cdot)_K$ and $(\nabla \cdot, \nabla \cdot)_K$.

Meanwhile, for every k, $1 \leqslant k \leqslant d$, we make an orthogonal decomposition of $P_M^{\mathcal{X}}(K)$ as
$$P_M^{\mathcal{X}}(K) = W_k(K) \oplus Y_k(K), \tag{2.15}$$
where $W_k(K)$ is perpendicular to $(q_{2k-1} - q_{2k})$, and $Y_k(K)$ is the orthogonal complementary of $W_k(K)$ with respect to the inner product $(\partial_k \cdot, \partial_k \cdot)_K$. Note that $\dim(\mathrm{span}\{(q_{2k-1} - q_{2k})\}) = 1$ and $(q_{2k-1} - q_{2k})$ is not orthogonal to $P_M^{\mathcal{X}}(K)$, we have obviously $\dim(Y_k(K)) = 1$. Denote by y_k the unique (up to a constant) basis function of $Y_k(K)$.

Now, given $\phi \in P_M^{\mathcal{X}}(K)$ and $\psi \in P_M^{\mathcal{F}}(K)$, they can be decomposed as
$$\phi = \phi_k' + \phi_k'', \quad \phi_k' \in \mathrm{span}\{y_k\} \text{ and } \phi_K'' \in W_k(K), \tag{2.16}$$
and
$$\psi = \sum_{k=1}^d \psi_k = \sum_{k=1}^d (\psi_k^{'} + \psi_k^{''}), \tag{2.17}$$
where $\psi_k \in Q_M^k(K)$, $\psi_k^{'} \in \mathrm{span}\{q_{2k-1} - q_{2k}\}$ and $\psi_k^{''} \in \mathrm{span}\{q_{2k-1} + q_{2k}\}$. Then we have
$$(\nabla \phi, \nabla \psi)_K = \sum_{k=1}^d (\nabla \phi, \nabla \psi_k)_K = \sum_{k=1}^d (\partial_k \phi, \partial_k \psi_k)_K = \sum_{k=1}^d (\partial_k \phi, \partial_k \psi_k')_K = \sum_{k=1}^d (\partial_k \phi_k', \partial_k \psi_k')_K.$$
Note that $\partial_k \phi_k' \not\equiv \partial_k \psi_k'$ unless both of them are zero; actually, the face average of $\partial_k y_k$ vanishes for every face, and the face average of $\partial_k (q_{2k-1} - q_{2k})$ does not vanish for F_{2k-1} or F_{2k}. We have
$$(\partial_k \phi_k', \partial_k \psi_k') \leqslant \theta_k (\|\partial_k \phi_k'\|_{0,K}^2 + \|\partial_k \psi_k'\|_{0,K}^2),$$
with $\theta_k < 1/2$ uniform on $\mathrm{span}\{y_k\} \times \mathrm{span}\{q_{2k-1} - q_{2k}\}$. Therefore,
$$\begin{aligned} |(\nabla \phi, \nabla \psi)_K| = |\sum_{k=1}^d (\partial_k \phi_k', \partial_k \psi_k')_K| &\leqslant \sum_{k=1}^d \theta_k (\|\partial_k \phi_k'\|_{0,K}^2 + \|\partial_k \psi_k'\|_{0,K}^2) \\ &\leqslant \left(\max_{1 \leqslant k \leqslant d} \theta_k \right) (\sum_{k=1}^d \|\partial_k \phi\|_{0,K}^2 + \sum_{k=1}^d \|\partial_k \psi_k\|_{0,K}^2) = \theta(\|\nabla \phi\|_{0,K}^2 + \|\nabla \psi\|_{0,K}^2), \end{aligned}$$
where $\theta := \max_{1 \leqslant k \leqslant d} \theta_k < 1/2$. This finishes the proof. $\square$

2.2.2 Property of the nodal interpolation

Lemma 2.4. For any $u \in P_3(K)$ and $v \in P_M(K)$, it holds that

$$(\nabla(u-\Pi_K u), \nabla v)_K = -\sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}} \frac{h_ih_j}{3}\int_K \frac{\partial^3 u}{\partial x_i\partial x_j^2}\frac{\partial v}{\partial x_i}\mathrm{d}x + \sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}} \frac{2}{45}h_i^3h_j\int_K \frac{\partial^3 u}{\partial x_i\partial x_j^2}\frac{\partial^3 v}{\partial x_i^3}\,\mathrm{d}x.$$

Proof. In order to simplify the calculations, we use auxiliary length $\xi_i = \frac{x_i - x_{i,c}}{h_i}, 1 \leqslant i \leqslant d$, when necessary. Since $u \in P_3(K)$, Taylor expansion yields

$$u - \Pi_K u = \sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}} \frac{h_i^2h_j}{2}\frac{\partial^3 u}{\partial x_i^2\partial x_j}\big(\xi_i^2\xi_j - \frac{4}{3}\xi_j + \frac{\xi_j^3}{3}\big). \tag{2.18}$$

Thus

$$\frac{\partial(u-\Pi_K u)}{\partial x_i} = \sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}} h_ih_j\frac{\partial^3 u}{\partial x_i^2\partial x_j}\xi_i\xi_j + \sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}} \frac{h_ih_j}{2}\frac{\partial^3 u}{\partial x_i\partial x_j^2}\big(\xi_j^2 - \frac{4}{3} + \xi_i^2\big). \tag{2.19}$$

It follows from the definition of $P_M(K)$ that

$$\frac{\partial v}{\partial x_i} = C_{1,i} + C_{2,i}h_i\xi_i + \sum_{\alpha\in M_i} C_{3,i}(\alpha)\big(\prod_{\substack{1\leqslant k\leqslant d\\ k\neq i}} h_k\xi_k^{\alpha_k}\big) + C_{4,i}h_i^2\xi_i^2, \tag{2.20}$$

where $C_{1,i}, C_{2,i}, C_{3,i}(\alpha), and C_{4,i}$ are all constant coefficients with respect to given v, and M_i is a set of multi-indices defined as $M_i := \{\alpha = (\alpha_1, \cdots, \alpha_d) | \alpha_i = 0,\ \alpha_k \in \{0,1\}, 1 \leqslant k \neq i \leqslant d, |\alpha| > 0\}$.

Elementary calculation yields

$$\int_K \big(h_ih_j\frac{\partial^3 u}{\partial x_i^2\partial x_j}\xi_i\xi_j\big)\frac{\partial v}{\partial x_i}\mathrm{d}x = 0,\ 1 \leqslant i \neq j \leqslant d, \tag{2.21}$$

$$\int_K \Big(\frac{h_ih_j}{2}\frac{\partial^3 u}{\partial x_i\partial x_j^2}\big(\xi_j^2 - \frac{4}{3} + \xi_i^2\big)\Big)\Big(C_{2,i}h_i\xi_i\Big)\mathrm{d}x = 0,\ 1 \leqslant i \neq j \leqslant d, \tag{2.22}$$

and

$$\int_K \Big(\frac{h_ih_j}{2}\frac{\partial^3 u}{\partial x_i\partial x_j^2}\big(\xi_j^2 - \frac{4}{3} + \xi_i^2\big)\Big)\Big(\sum_{\alpha\in M_i} C_{3,i}(\alpha)\big(\prod_{\substack{1\leqslant k\leqslant d\\ k\neq i}} h_k\xi_k^{\alpha_k}\big)\Big)\mathrm{d}x = 0,\ 1 \leqslant i \neq j \leqslant d. \tag{2.23}$$

So, we only need to calculate $C_{1,i}$ and $C_{4,i}$, which read

$$C_{1,i} = \frac{1}{|K|}\int_K \frac{\partial v}{\partial x_i}\,\mathrm{d}x - \frac{h_i^2}{3|K|}\int_K \frac{\partial^3 v}{\partial x_i^3}\,\mathrm{d}x, \quad \text{and} \quad C_{4,i} = \frac{1}{2}\frac{\partial^3 v}{\partial x_i^3}.$$

Elementary calculation yields

$$\begin{aligned}\int_K C_{1,i}\Big(\frac{h_ih_j}{2}\frac{\partial^3 u}{\partial x_i\partial x_j^2}\big(\xi_j^2 - \frac{4}{3} + \xi_i^2\big)\Big)\,\mathrm{d}x &= -\frac{C_{1,i}h_ih_j}{3}\int_K \frac{\partial^3 u}{\partial x_i\partial x_j^2}\,\mathrm{d}x \\ &= -\frac{h_ih_j}{3}\int_K \frac{\partial^3 u}{\partial x_i\partial x_j^2}\frac{\partial v}{\partial x_i}\mathrm{d}x + \frac{h_i^3h_j}{9}\int_K \frac{\partial^3 u}{\partial x_i\partial x_j^2}\frac{\partial^3 v}{\partial x_i^3}\,\mathrm{d}x,\ 1 \leqslant i \neq j \leqslant d,\end{aligned} \tag{2.24}$$

and

$$\int_K C_{4,i}h_i^2\xi_i^2\Big(\frac{h_ih_j}{2}\frac{\partial^3 u}{\partial x_i^2\partial x_j}\big(\xi_j^2 - \frac{4}{3} + \xi_i^2\big)\Big)\,\mathrm{d}x = -\frac{h_i^3h_j}{15}\int_K \frac{\partial^3 u}{\partial x_i^2\partial x_j}\frac{\partial^3 v}{\partial x_i^3}\mathrm{d}x,\ 1 \leqslant i \neq j \leqslant d. \tag{2.25}$$

A combination of (2.20) and (2.19) and some elementary calculation yield,

$$\int_K \frac{\partial(u-\Pi_K u)}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x = -\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}} \frac{h_ih_j}{3}\int_K \frac{\partial^3 u}{\partial x_i\partial x_j^2}\frac{\partial v}{\partial x_i}\mathrm{d}x + \sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}} \frac{2}{45}h_i^3h_j\int_K \frac{\partial^3 u}{\partial x_i\partial x_j^2}\frac{\partial^3 v}{\partial x_i^3}\,\mathrm{d}x. \tag{2.26}$$

A summation of (2.26) with respect to i from 1 to d completes the proof. $\square$

3 Rectangular Morley element scheme for second order problems

3.1 Subdivision of the domain and the finite element space

For simplicity, in this paper, let $\Omega \subset \mathbb{R}^d$ be a bounded domain which can be subdivided to a rectangular triangulation $\mathcal{T}_h$. For the triangulation $\mathcal{T}_h$, let $\mathcal{N}_h$ denote the set of all the vertices, $\mathcal{N}_h = \mathcal{N}_h^i \cup \mathcal{N}_h^b$, with $\mathcal{N}_h^i$ and $\mathcal{N}_h^b$ consisting of the interior vertices and the boundary vertices, respectively. Similarly, let $\mathcal{F}_h = \mathcal{F}_h^i \bigcup \mathcal{F}_h^b$ denote the set of all the $(d-1)$-faces, with $\mathcal{F}_h^i$ and $\mathcal{F}_h^b$ consisting of the interior faces and the boundary faces, respectively. For $f \in \mathcal{F}_h^i$, K_f^L and K_f^R are the two adjacent elements that share f as a common face, and $\mathbf{n}_f^L$ and $\mathbf{n}_f^R$ denote the unit outer normal vectors of K_f^L and K_f^R, respectively, on f. Given a triangulation $\mathcal{T}_h$ of Ω, define the d-rectangular Morley element space

$$V_h := \{v_h \in L^2(\Omega) :\ v_h|_K \in P_M(K),\ v_h(x) \text{ is continuous for } x \in \mathcal{N}_h,\ \int_f \frac{\partial v_h}{\partial \mathbf{n}_f} ds \text{ is continuous for } f \in \mathcal{F}_h^i\},$$

and associated with $H_0^1(\Omega)$

$$V_{h0} := \{v_h \in V_h : v_h(x) = 0,\ \text{for}\, x \in \mathcal{N}_h^b\}.$$

With respect to the vertices and the faces respectively, we define

$$V_h^{\mathcal{X}} := \{v_h \in V_h : \int_f \frac{\partial v_h}{\partial \mathbf{n}_f} ds = 0,\ \forall f \in \mathcal{F}_h\}, \quad \text{and} \quad V_h^{\mathcal{F}} := \{v_h \in V_h : v_h(x) = 0,\ \forall x \in \mathcal{N}_h\},$$

and for any $f \in \mathcal{F}_h$,

$$V_h^f := \{v_h \in V_h : v_h(x) = 0,\ \forall x \in \mathcal{N}_h;\ \int_{f'} \frac{\partial v_h}{\partial \mathbf{n}_{f'}} ds = 0,\ \forall f^{'} \in \mathcal{F}_h \text{ other than } f\}.$$

Also, define

$$V_{h0}^{\mathcal{X}} := V_h^{\mathcal{X}} \cap V_{h0}.$$

Evidently,

$$V_h = V_h^{\mathcal{F}} \oplus V_h^{\mathcal{X}} = \oplus_{f \in \mathcal{F}} V_h^f \oplus V_h^{\mathcal{X}},\ \text{and}\ V_{h0} = V_h^{\mathcal{F}} \oplus V_{h0}^{\mathcal{X}} = \oplus_{f \in \mathcal{F}} V_h^f \oplus V_{h0}^{\mathcal{X}}.$$

For each element $K \in \mathcal{T}_h$, let h_K be the diameter of the smallest ball containing K, and ρ_K be the diameter of the largest ball contained in K. Let $\mathcal{T}_h$ belong to a family of triangulations described in previous section with $h \to 0$. We assume that $\mathcal{T}_h$ satisfied that $h_K \leqslant h \leqslant \eta \rho_K$, $\forall K \in \mathcal{T}_h$ for a positive constant η independent of h.

We introduce the following triangulation-dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\|v\|_{m,h} = (\sum_{K \in \mathcal{T}_h} \|v\|_{m,K}^2)^{1/2}, |v|_{m,h} = (\sum_{K \in \mathcal{T}_h} |v|_{m,K}^2)^{1/2}$$

for such functions v that $v|_K \in H^m(K), \forall K \in \mathcal{T}_h$.

3.2 Model problem and its discretization

We consider the following second order elliptic problem: with $f \in L^2(\Omega)$,

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Δ is the standard Laplacian operator.

Define

$$a(u, v) = \int_\Omega \sum_{i=1}^d \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx.$$

The weak form of problem (3.1) is: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \ \forall v \in H_0^1(\Omega). \tag{3.2}$$

For $v, w \in L^2(\Omega)$ that $v|_K, w|_K \in H^1(K), \forall K \in \mathcal{T}_h$, we define

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx.$$

The finite element method for problem (3.1) is: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h), \ \forall v_h \in V_{h0}. \tag{3.3}$$

Because v_h is continuous on $\mathcal{N}_h$, the weak continuity property ensures the uniqueness of the solution.

3.3 Some intrinsic properties of V_h

3.3.1 Stable decomposition with respect to vertices and faces

Firstly, we show that the decomposition of V_h with respect to vertices and faces is stable.

Lemma 3.1. For any $v_h \in V_h$, there exist uniquely $v_h^{\mathcal{X}} \in V_h^{\mathcal{X}}$, $v_h^{\mathcal{F}} \in V_h^{\mathcal{F}}$, such that

$$v_h = v_h^{\mathcal{X}} + v_h^{\mathcal{F}},$$

and moreover,

$$|v_h^{\mathcal{X}}|_{1,h} + |v_h^{\mathcal{F}}|_{1,h} \lesssim |v_h|_{1,h}.$$

Proof. Given $v_h \in V_h$, the existence and uniqueness of $v_h^{\mathcal{X}}$ and $v_h^{\mathcal{F}}$ is evident. Now we prove the stability of the decomposition. On every cell K,

$$\begin{aligned}(\nabla_h v_h, \nabla_h v_h)_K &= (\nabla_h(v_h^{\mathcal{X}} + v_h^{\mathcal{F}}), \nabla_h(v_h^{\mathcal{X}} + v_h^{\mathcal{F}}))_K \\ &= (\nabla_h v_h^{\mathcal{X}}, \nabla_h v_h^{\mathcal{X}})_K + (\nabla_h v_h^{\mathcal{F}}, \nabla_h v_h^{\mathcal{F}})_K + 2(\nabla_h v_h^{\mathcal{X}}, \nabla_h v_h^{\mathcal{F}})_K.\end{aligned}$$

By Lemma 2.3,

$$|(\nabla_h v_h^{\mathcal{X}}, \nabla_h v_h^{\mathcal{F}})_K| \leqslant \theta((\nabla_h v_h^{\mathcal{X}}, \nabla_h v_h^{\mathcal{X}})_K + (\nabla_h v_h^{\mathcal{F}}, \nabla_h v_h^{\mathcal{F}})_K),$$

therefore,

$$(\nabla_h v_h, \nabla_h v_h)_K \geqslant (1-2\theta)((\nabla_h v_h^{\mathcal{X}}, \nabla_h v_h^{\mathcal{X}})_K + (\nabla_h v_h^{\mathcal{F}}, \nabla_h v_h^{\mathcal{F}})_K).$$

Making a summation on every cell K, we obtain that

$$(\nabla_h v_h, \nabla_h v_h) \geqslant (1-2\theta)((\nabla_h v_h^{\mathcal{X}}, \nabla_h v_h^{\mathcal{X}}) + (\nabla_h v_h^{\mathcal{F}}, \nabla_h v_h^{\mathcal{F}})).$$

Then we have

$$\|\nabla_h v_h\|_{0,\Omega} \geqslant (\frac{1}{2} - \theta)(\|\nabla_h v_h^{\mathcal{X}}\|_{0,\Omega} + \|\nabla_h v_h^{\mathcal{F}}\|_{0,\Omega}).$$

This finishes the proof. □

3.3.2 The essential continuity of the finite element functions

In this section, we estimate the consistency error with respect to different finite element functions.

Lemma 3.2. The estimate below holds uniformly for any shape-regular family of triangulations.

1. It holds for $v_h^{\mathcal{X}} \in V_{h0}^{\mathcal{X}}$ that

$$|a_h(v, v_h^{\mathcal{X}}) + (\Delta v, v_h^{\mathcal{X}})| \lesssim h^{k-1}|v|_{k,\Omega}|v_h^{\mathcal{X}}|_{1,h}, \ \forall v \in H^k(\Omega), \ k=2,3; \tag{3.4}$$

2. It holds for $v_h^{\mathcal{F}} \in V_h^{\mathcal{F}}$ that

$$|a_h(v, v_h^{\mathcal{F}}) + (\Delta v, v_h^{\mathcal{F}})| \lesssim h|v|_{2,\Omega}|v_h^{\mathcal{F}}|_{1,h},\ \forall v \in H^2(\Omega). \tag{3.5}$$

Proof. The assertions fall into the standard finite element analysis. On every element K, let $P_K^0 : L^2(K) \to P_0(K)$, $P_K^1 : L^2(K) \to P_1(K)$ be the L^2 orthogonal projection. For the first item, we have

$$\begin{aligned}
|a_h(v, v_h^{\mathcal{X}}) + (\Delta v, v_h^{\mathcal{X}})| &= |\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\int_{\partial K}\frac{\partial v}{\partial x_i}v_h^{\mathcal{X}}n_i ds| = |\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\int_{\partial K}\frac{\partial v}{\partial x_i}(v_h^{\mathcal{X}} - \Pi_K^1 v_h^{\mathcal{X}})n_i ds| \\
\text{(by Lemma 2.1)} \quad &= |\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\int_{\partial K}(\frac{\partial v}{\partial x_i} - P_K^{k-2}\frac{\partial v}{\partial x_i})(v_h^{\mathcal{X}} - \Pi_K^1 v_h^{\mathcal{X}})n_i ds| \\
&\leqslant \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\|\frac{\partial v}{\partial x_i} - P_K^{k-2}\frac{\partial v}{\partial x_i}\|_{0,\partial K}\|v_h^{\mathcal{X}} - \Pi_K^1 v_h^{\mathcal{X}}\|_{0,\partial K} \lesssim \sum_{K\in\mathcal{T}_h} h_K^k|v|_{k,K}|v_h^{\mathcal{X}}|_{2,K} \\
&\lesssim h^{k-1}|v|_{k,\Omega}|v_h^{\mathcal{X}}|_{1,h},\ k = 2, 3.
\end{aligned}$$

By the similar technique, for any $(d-1)$-face f, making use of Lemma 2.1-(2), we obtain (3.5). The proof is completed. □

Lemma 3.3. It holds for all such f that ω_f is uniform that

$$|a_{\omega_f}(v, v_h^f) + (\Delta v, v_h^f)_{\omega_f}| \lesssim h_{\omega_f}^2|v|_{3,\omega_f}|v_h^f|_{1,\omega_f},\ \forall\ v_h^f \in P_M^f(\omega_f),\ v \in H^3(\omega_f). \tag{3.6}$$

Here $a_{\omega_f}(v, v_h^f) = (\nabla v, \nabla v_h^f)_{K_L} + (\nabla v, \nabla v_h^f)_{K_R}$.

Proof. By Lemma 2.2, the lemma follows by the same technique as that for Lemma 3.2. □

Lemma 3.4. For any d-rectangle K, and $f \subset \partial K$ a $(d-1)$-face, it holds that

$$|(\nabla v, \nabla v_h^f)_K + (\Delta v, v_h^f)_K| \lesssim h_K^2|v|_{3,K}|v_h^f|_{1,K},\ \forall\ v_h^f \in P_M^f(K),\ v \in H^3(K),\ v|_f = 0. \tag{3.7}$$

Proof. Without loss of generality, assume the normal direction of f to be x_1, and $f = F_1$. From the expression of the basis function, $v_h^f = 0$ on F_1 and F_2. Using formula of integration by parts, we have

$$\begin{aligned}
|(\nabla v, \nabla v_h^f)_K \quad &+ \quad (\Delta v, v_h^f)_K| = |\sum_{i=1}^{d}\int_{\partial K}\frac{\partial v}{\partial x_i}v_h^f n_i ds| \\
&= |\sum_{i=2}^{d}(\int_{F_{2i-1}}\frac{\partial v}{\partial x_i}v_h^f ds - \int_{F_{2i}}\frac{\partial v}{\partial x_i}v_h^f ds)| = |\sum_{i=2}^{d}\int_K\frac{\partial^2 v}{\partial x_i^2}v_h^f dx|
\end{aligned}$$

Because $v|_f = 0$, $\frac{\partial^2 v}{\partial x_k^2} = 0 (2 \leqslant k \leqslant d)$ on f. Then by Poincaré inequality, we have $|\frac{\partial^2 v}{\partial x_k^2}|_{0,K} \lesssim h_K|v|_{3,K}$. Because $\Pi_K^1 v_h^f = 0$,

$$\begin{aligned}
|\sum_{i=2}^{d}\int_K\frac{\partial^2 v}{\partial x_i^2}v_h^f dx| &= |\sum_{i=2}^{d}\int_K\frac{\partial^2 v}{\partial x_i^2}(v_h^f - \Pi_K^1 v_h^f)dx| \\
&\lesssim \sum_{i=2}^{d}\|\frac{\partial^2 v}{\partial x_i^2}\|_{0,K}\|v_h^f - \Pi_K^1 v_h^f\|_{0,K} \lesssim h_K^2|v|_{3,K}|v_h^f|_{1,K}.
\end{aligned}$$

This finishes the proof. □

Lemma 3.5. The estimate below holds for any shape-regular family of uniform triangulations:

$$|a_h(v, v_h^{\mathcal{F}}) + (\Delta v, v_h^{\mathcal{F}})| \lesssim h^2|v|_{3,\Omega}|v_h^{\mathcal{F}}|_{1,h},\ \forall v \in H^3(\Omega) \cap H_0^1(\Omega),\ v_h \in V_h^{\mathcal{F}}. \tag{3.8}$$

Proof. Given $v_h^{\mathcal{F}} \in V_h^{\mathcal{F}}$, a decomposition follows that $v_h^{\mathcal{F}} = \sum_{f\in\mathcal{F}_h} v_h^f$, with $v_h^f \in V_h^f$. By Lemma 3.3 and Lemma 3.4, we obtain

$$\begin{aligned} |a_h(v, v_h^{\mathcal{F}}) + (\Delta v, v_h^{\mathcal{F}})| &\leqslant \sum_{f\in\mathcal{F}_h} |a_h(v, v_h^f) + (\Delta v, v_h^f)| \\ &\lesssim h^2 \Big(\sum_{f\in\mathcal{F}_h^i} |v|_{3,\omega_f} |v_h^f|_{\omega_f} + \sum_{f\in\mathcal{F}_h^b} |v|_{3,K_f} |v_h^f|_{K_f}\Big) \\ &\lesssim h^2 \Big(\sum_{f\in\mathcal{F}_h^i} |v|_{3,\omega_f}^2 + \sum_{f\in\mathcal{F}_h^b} |v|_{3,K_f}^2\Big)^{1/2} \Big(\sum_{f\in\mathcal{F}_h} |v_h^f|_{1,h}^2\Big)^{1/2}. \end{aligned}$$

Here, K_f is the element containing $(d-1)$-face f for $f \subset \partial\Omega$. Evidently, $(\sum_{f\in\mathcal{F}_h^i} |v|_{3,\omega_f}^2 + \sum_{f\in\mathcal{F}_h^b} |v|_{3,K_f}^2)^{1/2} \lesssim |v|_{3,\Omega}$. Besides, by Lemma 2.3,

$$\sum_{f\in\mathcal{F}_h} |v_h^f|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_h} \|\nabla v_h^f\|_{0,K}^2 \lesssim \sum_{K\in\mathcal{T}_h} \|\nabla \sum_{f\in\mathcal{F}_h} v_h^f\|_{0,K}^2 = \sum_{K\in\mathcal{T}_h} \|\nabla v_h^{\mathcal{F}}\|_{0,K}^2. \tag{3.9}$$

Therefore, (3.8) follows. This finishes the proof. □

Lemma 3.6. Given $v_h \in V_h$, there exist uniquely $v_h^{\mathcal{X}} \in V_h^{\mathcal{X}}$ and $v_h^f \in V_h^f (f \in \mathcal{F}_h)$, so that

$$v_h = v_h^{\mathcal{X}} + \sum_{f\in\mathcal{F}_h} v_h^f, \tag{3.10}$$

and moreover,

$$|v_h^{\mathcal{X}}|_{1,h}^2 + \sum_{f\in\mathcal{F}_h} |v_h^f|_{1,h}^2 \lesssim |v_h|_{1,h}^2. \tag{3.11}$$

Proof. The proof follows from Lemma 3.1 and (3.9). □

This lemma can hint an optimal fast auxiliary space preconditioner for the discretized system.

3.4 Convergence Analysis: error estimate in energy norm

To estimate the convergence rate of the discretization, we begin with the famous Strang lemma below.

Lemma 3.7. (the second Strang Lemma) Let u and u_h be the solutions of problems (3.1) and (3.2) respectively. Then

$$|u - u_h|_{1,h} \eqsim \inf_{v_h\in V_{h0}} |u - v_h|_{1,h} + \sup_{0\neq w_h\in V_{h0}} \frac{|(f, w_h) - a_h(u, w_h)|}{|w_h|_{1,h}}.$$

Lemma 3.8. Let V_h and V_{h0} be the finite element spaces of the d-rectangular Morley element. Then, for $k = 2, 3$, we have

$$\inf_{v_h\in V_h} \sum_{m=0}^{k} h^m |v - v_h|_{m,h} \lesssim h^k |v|_k,\ \forall v \in H^k(\Omega),$$

$$\inf_{v_h\in V_{h0}} \sum_{m=0}^{k} h^m |v - v_h|_{m,h} \lesssim h^k |v|_k,\ \forall v \in H^k(\Omega) \cap H_0^1(\Omega).$$

Theorem 3.9. Let u and u_h be the solutions of problems (3.1) and (3.2) respectively. Then

1. if $u \in H_0^1(\Omega) \cap H^2(\Omega)$, then
$$|u - u_h|_{1,h} \lesssim h|u|_2. \tag{3.12}$$

2. if the triangulation is uniform, and $u \in H_0^1(\Omega) \cap H^3(\Omega)$, then
$$|u - u_h|_{1,h} \lesssim h^2|u|_3. \tag{3.13}$$

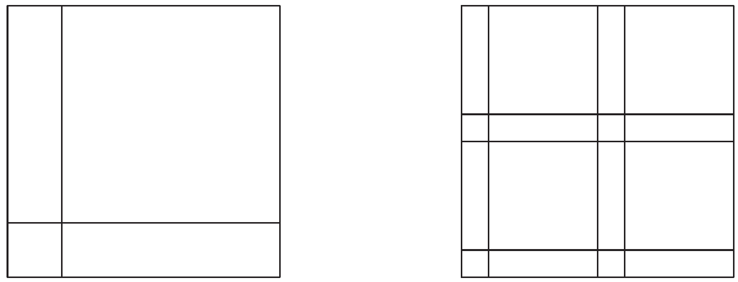

Figure 2. Illustration of a general shape regular triangulations in two dimensions. The triangulation in right is a combination of small patterns as the left one.

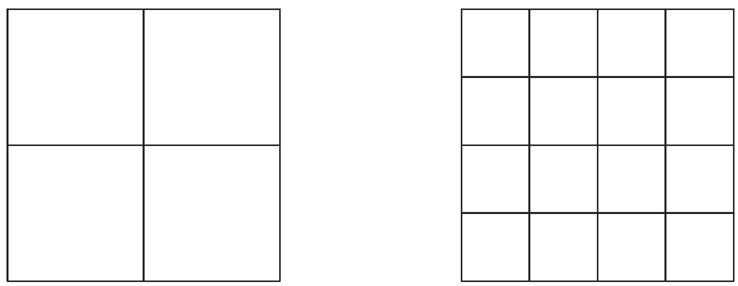

Figure 3 Illustration of uniform triangulations in two dimensions.

Proof. By Strang lemma and the approximation estimate Lemma 3.8, we only have to study the consistency error, which can be estimated by Lemma 3.2, Lemma 3.5 and the stability of decomposition of Lemma 3.1. The proof is finished. □

Theorem 3.9 reveals that generally on a shape-regular family of triangulations, which can be as "bad" as ones shown in Figure 2, the error decays with $\mathcal{O}(h)$ order in energy norm, and on a family of triangulation, like one shown in Figure 3, an $\mathcal{O}(h^2)$ order can be expected. There is also an intervenient result on **divisionally uniform** triangulation. For a family of divisionally uniform triangulations, we refer to a family of conforming triangulations on Ω which are uniform triangulations on Ω_j, $j=1,2,\dots,J$, with $\{\Omega_j\}_{j=1:J}$ a subdivision of Ω. (Figure 4.)

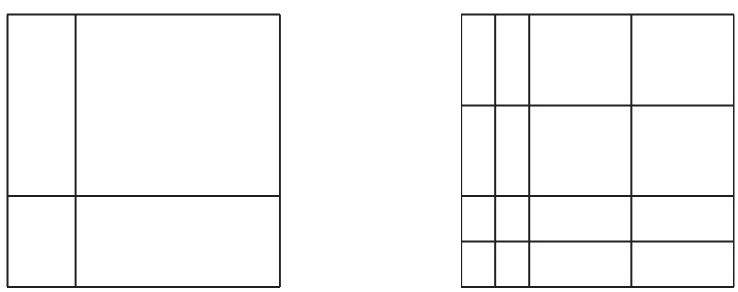

Figure 4. Illustration of divisionally uniform triangulations in two dimension. As shown in the left figure, the domain consists of four subdomains, and restricted in each subdomain, the triangulation is uniform.

Theorem 3.10. Let u and u_h be the solutions of problems (3.1) and (3.2) respectively. If the triangulation is divisionally uniform, and $u\in H^1_0(\Omega)\cap H^{2.5}(\Omega)$,

$$|u-u_h|_{1,h}\lesssim h^{1.5}\|u\|_{2.5}. \tag{3.14}$$

For simplicity, we only prove in detail the case that Ω is divided to two subdomains. The more complicated cases are on the same line.

Let $\Omega = \Omega_1 \cup \Omega_2$, and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ be the interface between the two subdomains. A stripe along the interface is denoted by

$$\Omega_\delta = \{x \in \Omega | \mathrm{dist}(x, \Gamma) \leqslant \delta\}.$$

The lemma below is important in the technical analysis.

Lemma 3.11. [2] For $v \in H^s(\Omega)$, where $0 \leqslant s \leqslant 0.5$, we have:

$$||v||_{0,\Omega_\delta} \lesssim \delta^s ||v||_s.$$

Lemma 3.12. On divisionally uniform triangulations, it holds for $v_h^{\mathcal{F}} \in V_h^{\mathcal{F}}$ that

$$|a_h(v, v_h^{\mathcal{F}}) + (\Delta v, v_h^{\mathcal{F}})_\Omega| \lesssim h^{1.5} \|v\|_{2.5} |v_h^{\mathcal{F}}|_{1,h}, \ v \in H^{2.5}(\Omega) \cap H_0^1(\Omega). \tag{3.15}$$

Proof. Given $v_h^{\mathcal{F}} \in V_h^{\mathcal{F}}$, it can be decomposed to $v_h = \sum_{f \in \mathcal{F}_h} v_h^f$. Thus

$$a_h(v, v_h^{\mathcal{F}}) + (\Delta v, v_h^{\mathcal{F}}) = \left(\sum_{f \subset \overline{\Omega_1} \setminus \Gamma} + \sum_{f \subset \overline{\Omega_2} \setminus \Gamma} + \sum_{f \subset \Gamma} \right) \left[a_h(v, v_h^f) + (\Delta v, v_h^f) \right].$$

It can be proved that

$$\left| \sum_{f \subset \overline{\Omega_1} \setminus \Gamma} \left[a_h(v, v_h^f) + (\Delta v, v_h^f) \right] \right| \lesssim h^{k-1} |v|_{k,\Omega_1} \| \sum_{f \subset \overline{\Omega_1} \setminus \Gamma} v_h^f \|_{1,h}, \quad k = 2, 3, \tag{3.16}$$

$$\left| \sum_{f \subset \overline{\Omega_2} \setminus \Gamma} \left[a_h(v, v_h^f) + (\Delta v, v_h^f) \right] \right| \lesssim h^{k-1} |v|_{k,\Omega_2} \| \sum_{f \subset \overline{\Omega_2} \setminus \Gamma} v_h^f \|_{1,h}, \quad k = 2, 3, \tag{3.17}$$

and

$$\left| \sum_{f \subset \Gamma} \left[a_h(v, v_h^f) + (\Delta v, v_h^f) \right] \right| \lesssim h |v|_{2, \bigcup_{f \subset \Gamma} \omega_f} \| \sum_{f \subset \overline{\Omega_2} \setminus \Gamma} v_h^f \|_{1,h}. \tag{3.18}$$

Evidently, $\cup_{f \subset \Gamma} \omega_f \subset \omega_H$, where H is the size of the biggest cell in $\mathcal{T}_h$, and $H \lesssim h$. Therefore, by Lemma 3.11,

$$|v|_{2, \bigcup_{f \subset \Gamma} \omega_f} \lesssim h^{0.5} \|v\|_{2.5}.$$

Combining (3.16), (3.17) and (3.18) leads to (3.15). This finishes the proof. □

Proof of Theorem 3.10 The proof follows just the same line as the proof of Theorem 3.9 provided Lemma 3.12. □

3.5 Convergence analysis: error estimate in L^2 norm

By standard duality argument, we can prove the upper bound of the L^2 norm of error on convex domains.

Theorem 3.13. Assume Ω is convex, let $f \in L^2(\Omega)$, and let u and u_h be the solutions of problems (3.1) and (3.2), respectively. Then

$$||u - u_h||_0 \lesssim h^2 \|f\|_0. \tag{3.19}$$

In general, the estimate above can not be improved. Assuming triangulationes are uniform, we present in detail the lower bound estimate of the error in L^2 norm. It follows the idea of [16]. The main result of this section is the theorem below.

Theorem 3.14. Let u and u_h be solutions of problem (3.1) and (3.2), respectively. Suppose that $u \in H_0^1(\Omega) \bigcap H^s(\Omega)$, $s \geqslant 3$ and $s > \frac{d}{2} + 1$. Then, provided $||f||_0 \neq 0$,

$$||u - u_h||_0 \geqslant \beta h^2, \tag{3.20}$$

where $\beta = \frac{\delta}{||f||_0}$, δ is a positive constant, which is independent of the triangulation size h and the triangulation size is small enough.

Remark 3.15. By the embedding theorem of the Sobolev space, we need higher regularity of the solution in higher dimensions in order to guarantee $H^s(K) \subset C^1(K)$. Furthermore, it ensures the continuity of interpolation operators.

Remark 3.16. For the rectangular domain Ω, the condition $||f||_0 \neq 0$ implies that $||\frac{\partial^2 u}{\partial x_i \partial x_j}||_0 \neq 0$, $1 \leqslant i \neq j \leqslant d$. In fact, if $||\frac{\partial^2 u}{\partial x_i \partial x_j}||_0 = 0$, $1 \leqslant i \neq j \leqslant d$, then u is of the form $u = \sum_{i=1}^d f(x_i)$, for some function $f(x_i)$ with respect to x_i. Then the boundary condition indicates $u \equiv 0$.

We postpone the proof of Theorem 3.14 after several technical lemmas.

Firstly, define the global interpolation operator Π_h and P^k to V_h by

$$\Pi_h|_K = \Pi_K \text{ for any } K \in \mathcal{T}_h. \tag{3.21}$$

and

$$P^k|_K = P^k_K \text{ for any } K \in \mathcal{T}_h. \tag{3.22}$$

By means of Lemma 2.4, we can obtain the following crucial result.

Lemma 3.17. For $u \in H^1_0(\Omega) \bigcap H^s(\Omega)$, $s \geqslant 3$ and $s > \frac{d}{2} + 1$, it holds that,

$$(\nabla_h(u - \Pi_h u), \nabla_h \Pi_h u) \geqslant \alpha h^2, \tag{3.23}$$

for some positive constant α, which is independent of the triangulation size h provided that $||f||_0 \neq 0$ and that the triangulation size is small enough.

Proof. Given any element K, we follow the idea of [16] to define $P_K v \in P_3(K)$ by

$$\int_K \nabla^l P_K v \,\mathrm{d}x = \int_K \nabla^l v \,\mathrm{d}x, \ l = 0, 1, 2, 3, \tag{3.24}$$

for any $v \in H^s(\Omega)$, ($s \geqslant 3$ and $s > \frac{d}{2} + 1$). Note that the operator P_K is well-defined. The interpolation operator P_K has the following error estimates:

$$\begin{aligned} &|v - P_K v|_{j,K} \lesssim h^{3-j}|v|_{3,K}, \ j = 0, 1, 2, 3,\\ &|v - P_K v|_{j,K} \lesssim h|v|_{j+1,K}, \ j = 0, 1, 2, \end{aligned} \tag{3.25}$$

provided that $v \in H^s(\Omega)$, ($s \geqslant 3$ and $s > \frac{d}{2} + 1$). It follows from the definition of P_K in (3.24) that

$$\nabla^3 P_K v = P^0_K \nabla^3 v. \tag{3.26}$$

By the aid of P_K, we have the following decomposition

$$\begin{aligned} (\nabla_h(u - \Pi_h u), \nabla_h \Pi_h u) &= \sum_{K \in \mathcal{T}_h} (\nabla_h(P_K u - \Pi_K P_K u), \nabla_h \Pi_K u)_K \\ &\quad + \sum_{K \in \mathcal{T}_h} (\nabla_h(Id - \Pi_K)(Id - P_K)u, \nabla_h \Pi_K u)_K \\ &= J_1 + J_2. \end{aligned} \tag{3.27}$$

By means of Lemma 2.4, the first term J_1 on the right-hand side of (3.27) can be rewritten as

$$\begin{aligned} J_1 = &- \sum_{K \in \mathcal{T}_h} \sum_{i=1}^d \sum_{\substack{1 \leqslant j \leqslant d\\ j \neq i}} \frac{h_i h_j}{3} \int_K \frac{\partial^3 P_K u}{\partial x_i \partial x_j^2} \frac{\partial \Pi_K u}{\partial x_i} \mathrm{d}x + \sum_{K \in \mathcal{T}_h} \sum_{i=1}^d \sum_{\substack{1 \leqslant j \leqslant d\\ j \neq i}} \frac{2}{45} h_i^3 h_j \int_K \frac{\partial^3 P_K u}{\partial x_i \partial x_j^2} \frac{\partial^3 \Pi_K u}{\partial x_i^3} \,\mathrm{d}x \\ = &- \sum_{K \in \mathcal{T}_h} \sum_{i=1}^d \sum_{\substack{1 \leqslant j \leqslant d\\ j \neq i}} \frac{h_i h_j}{3} \int_K \frac{\partial^3 u}{\partial x_i x_j^2} \frac{\partial u}{\partial x_i} \mathrm{d}x + \sum_{K \in \mathcal{T}_h} \sum_{i=1}^d \sum_{\substack{1 \leqslant j \leqslant d\\ j \neq i}} \frac{h_i h_j}{3} \int_K \frac{\partial^3 (Id - P_K)u}{\partial x_i \partial x_j^2} \frac{\partial \Pi_K u}{\partial x_i} \,\mathrm{d}x \\ &+ \sum_{K \in \mathcal{T}_h} \sum_{i=1}^d \sum_{\substack{1 \leqslant j \leqslant d\\ j \neq i}} \frac{h_i h_j}{3} \int_K \frac{\partial^3 u}{\partial x_i \partial x_j^2} \frac{\partial (Id - \Pi_K)u}{\partial x_i} \mathrm{d}x + \sum_{K \in \mathcal{T}_h} \sum_{i=1}^d \sum_{\substack{1 \leqslant j \leqslant d\\ j \neq i}} \frac{2}{45} h_i^3 h_j \int_K \frac{\partial^3 P_K u}{\partial x_i \partial x_j^2} \frac{\partial^3 \Pi_K u}{\partial x_i^3} \,\mathrm{d}x. \end{aligned}$$

Since the triangulation is uniform and the boundary condition $u = 0$, on $\partial\Omega$, thus $\frac{\partial u}{\partial x_i}|_{\Gamma_{x_j}} = 0$, where Γ_{x_j} is the face of $\partial\Omega$ perpendicular to x_j axe and $j \neq i$, integrating by parts yields

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}}\frac{h_ih_j}{3}\int_K\frac{\partial^3 u}{\partial x_i\partial x_j^2}\frac{\partial u}{\partial x_i}\mathrm{d}x\\
&=-\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}}\frac{h_ih_j}{3}\int_K\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\Big)^2\mathrm{d}x+\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}}\frac{h_ih_j}{3}\int_{\Gamma_{x_j}}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}\mathrm{d}s\\
&=-\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}}\frac{h_ih_j}{3}\Big|\Big|\frac{\partial^2 u}{\partial x_i\partial x_j}\Big|\Big|_{0,K}^2.
\end{aligned}$$

By the commuting property of (3.26),

$$\frac{\partial^3 (Id-P_K)u}{\partial x_i\partial x_j^2}=(Id-P_K^0)\frac{\partial^3 u}{\partial x_i\partial x_j^2},\ 1\leqslant i\neq j\leqslant d.$$

Note that

$$\sum_{i=1}^{d}\Big|\Big|\frac{\partial \Pi_K u}{\partial x_i}\Big|\Big|_{0,K}\lesssim |u|_{3,K}.$$

This and the error estimate of (3.25) yield

$$J_1=\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{d}\sum_{\substack{1\leqslant j\leqslant d\\ j\neq i}}\frac{h_ih_j}{3}\Big|\Big|\frac{\partial^2 u}{\partial x_i\partial x_j}\Big|\Big|_{0,K}^2+O(h^2)||(Id-P_K^0)\nabla_h^3 u||_{0,K}|u|_{3,K}. \tag{3.28}$$

We turn to the second term J_2 on the right-hand side of (3.27). By the Poincaré inequality, and the commuting property of (3.26),

$$|J_2| \ \lesssim\ h^2\sum_{K\in\mathcal{T}_h}||\nabla_h^3(Id-P_K)u||_{0,K}|u|_3\lesssim h^2||(Id-P^0)\nabla_h^3 u||_0|u|_3. \tag{3.29}$$

Since the piecewise constant functions are dense in the space $L^2(\Omega)$,

$$||(Id-P^0)\nabla^3 u||_0\to\ 0,\ \ when\ \ h\to\ 0. \tag{3.30}$$

Summation of (3.28), (3.29) and (3.30) completes the proof. □

Again, the lemma below can be found in [17].

Lemma 3.18. Let u and u_h be solutions of problem (3.1) and (3.2), respectively. Then,

$$\begin{aligned}
(-f,u-u_h)=a_h(u,\Pi_h u-u_h)-(f,\Pi_h u-u_h)\\
+a_h(u-\Pi_h u,u-\Pi_h u)+a_h(u-\Pi_h u,u_h-\Pi_h u)\\
+2(f,\Pi_h u-u)+2a_h(u-\Pi_h u,\Pi_h u).
\end{aligned} \tag{3.31}$$

Proof of Theorem 3.14 It follows from Lemma 3.2 that

$$a_h(u,\Pi_h u-u_h)-(f,\Pi_h u-u_h)\lesssim h^2|u|_3|\Pi_h u-u_h|_h\lesssim h^4|u|_3^2. \tag{3.32}$$

By the Cauchy-Schwarz inequality and the error estimate Theorem 3.8, it yields

$$a_h(u-\Pi_h u,u-\Pi_h u)+2(f,\Pi_h u-u)\lesssim h^4(|u|_3+||f||_0)|u|_3, \tag{3.33}$$

$$a_h(u-\Pi_h u,u_h-\Pi_h u)\lesssim h^4|u|_3^2. \tag{3.34}$$

The error estimate of the last term of (3.31) by Lemma 3.17 gives

$$\alpha h^2 \leqslant a_h(u - \Pi_h u, \Pi_h u). \tag{3.35}$$

Hence, a combination of (3.31) - (3.35) leads to

$$(-f, u - u_h) \geqslant \delta h^2. \tag{3.36}$$

for some positive constant δ, which is independent of the triangulation size h and the triangulation size is small enough.

Therefore,

$$||u - u_h||_0 = \sup_{0 \neq w \in L^2(\Omega)} \frac{(w, u - u_h)}{||w||_0} \geqslant \frac{(-f, u - u_h)}{||-f||_0} \geqslant \frac{\delta}{||f||_0} h^2. \tag{3.37}$$

This finishes the proof. □

4 Numerical example

In this section, we present some numerical results of the two-dimensional and three-dimensional RM element by uniform triangulation, divisionally uniform triangulation, and general shape regular triangulation, respectively, of domain Ω to demonstrate our theoretical results. We follow the approaches shown in Figures 4, 3 and 2 to generate divisionally uniform triangulations, uniform triangulations, and general shape regular triangulations, respectively.

4.1 Two-dimensional examples

For two-dimensional experiments, we choose the computation domain to be $\Omega = [0, 1]^2$. We choose f such that the exact solution is $u_1(x, y) = x(1 - x)y(1 - y)$ and $u_2(x, y) = \sin(\pi x)\sin(\pi y)$, respectively. We run the numerical experiments with respect to different kinds of triangulations, and record the convergence rate in Figures 5 (for uniform triangulations), 6 (for divisionally uniform triangulations), and 7 (for general shape regular triangulations), respectively.

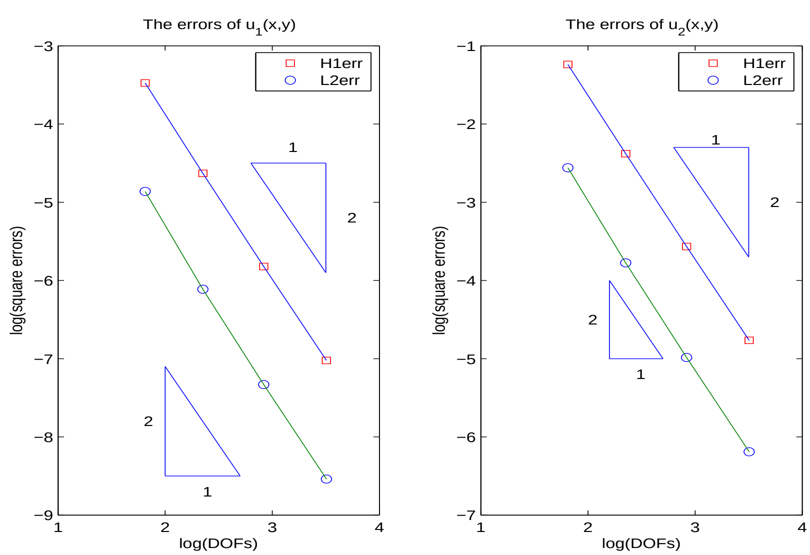

Figure 5 The errors in L^2 and H^1 norms with respect to $u_1(x, y)$ and $u_2(x, y)$ on uniform triangulations.

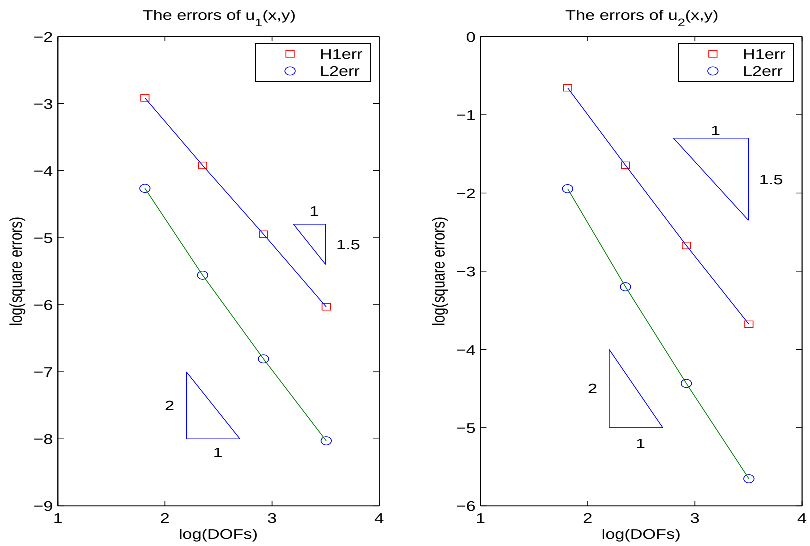

Figure 6 The errors in L^2 and H^1 norms with respect to $u_1(x,y)$ and $u_2(x,y)$ on divisionally uniform triangulations.

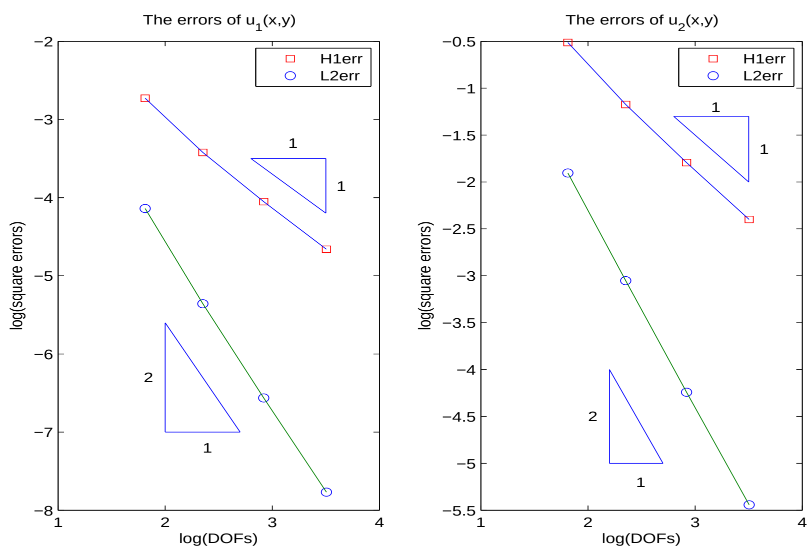

Figure 7 The errors in L^2 and H^1 norms with respect to $u_1(x,y)$ and $u_2(x,y)$ on general shape regular triangulations.

4.2 Three-dimensional examples

For three-dimensional experiments, we choose the computation domain to be In $\Omega = [0,1]^3$. We choose f such that the exact solution is $u_1(x,y,z) = x(1-x)y(1-y)z(1-z)$ and $u_2(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$, respectively. We run the numerical experiments with respect to different kinds of triangulations, and record the convergence rate in Figures 8 (for uniform triangulations), 9 (for divisionally uniform triangulations), and 10 (for general shape regular triangulations), respectively.

5 Concluding remarks

In this paper, we present a rigorous analysis of the RM element applied for second order problem in arbitrary dimensions. To be combined with the standard framework, some special properties of the RM

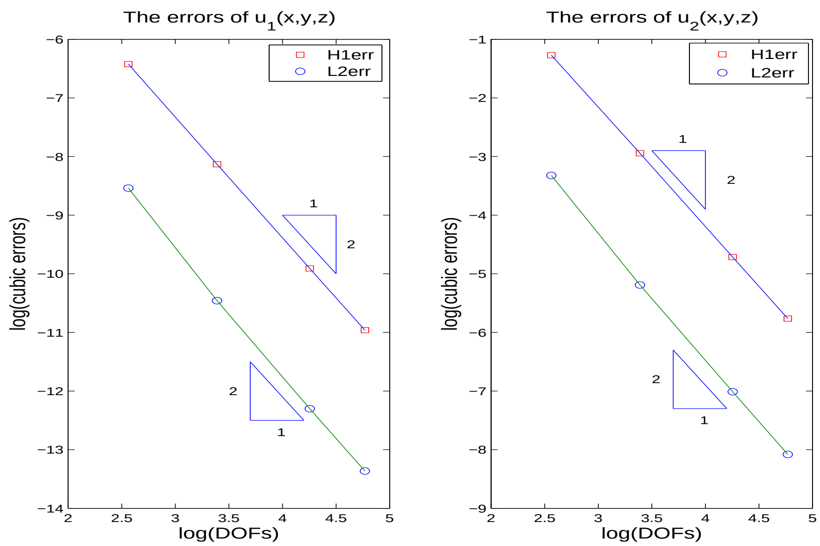

Figure 8 The errors in L^2 and H^1 norms with respect to $u_1(x,y,z)$ and $u_2(x,y,z)$ on uniform triangulations.

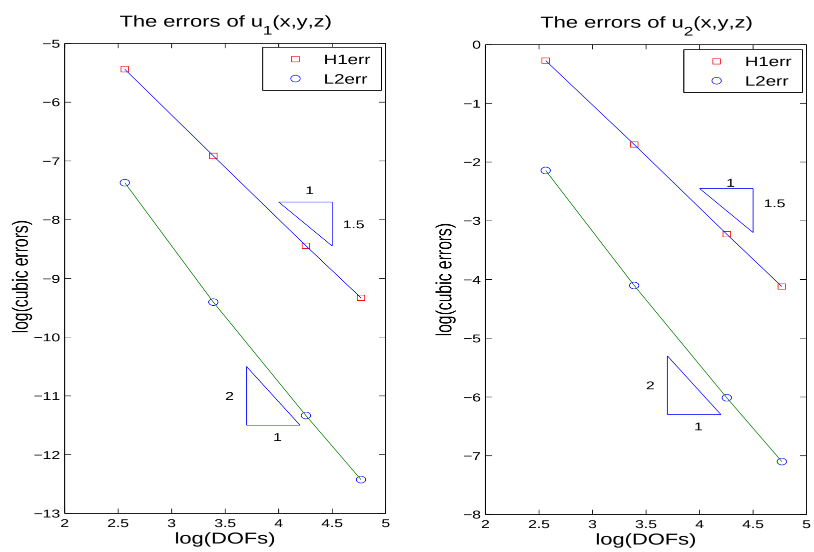

Figure 9. The errors in L^2 and H^1 norms with respect to $u_1(x,y,z)$ and $u_2(x,y,z)$ on divisionally uniform triangulations.

element functions are revealed and used. Both the energy norm and the L^2 norm of the error are studied, and the upper bound and the lower bound obtained illustrate that the analysis presented here is optimal. The RM element pretends to be one fit for fourth order elliptic perturbation problems, and can also expect application for contact/obstacle problems (c.f., e.g., [34,38]) in the future.

The fundamental role of stable decomposition for implementing the "divide and conquer" strategy is corroborated again in the analysis of the error estimate in energy norm. Also, the stable decomposition as revealed by Lemma 3.6 can be used to design an optimal preconditioner under the framework of fast auxiliary space preconditioning [13,46,47,54]. Besides, it is quite interesting to note that the RM element space does not contain a nontrivial conforming subspace. This unusual fact makes the *a posteriori* error analysis of the scheme a problem which absorbs theoretical interests. These will be studied in future works.

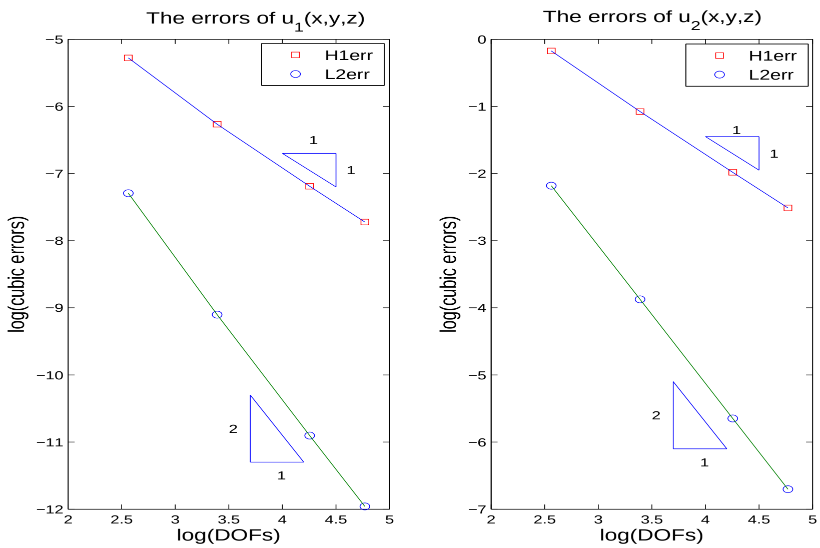

Figure 10. The errors in L^2 and H^1 norms with respect to $u_1(x,y,z)$ and $u_2(x,y,z)$ on general shape regular triangulations.

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